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Introduction to baricentric geometry with applications

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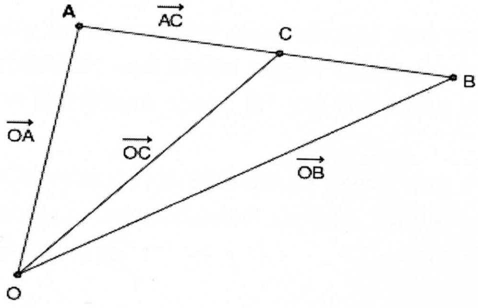
ABSTRACT. In this paper we present some interesting applications of the baricentric geometry.

MAIN RESULTS

Some preliminary facts. First recall that any two non collinear vectors $\overrightarrow{OA}, \overrightarrow{OB}$ create a basis on the plane with origin O, that is for any vector $\overrightarrow{OC}$ there are unique $p, q \in \mathbb{R}$ such that

$$\overrightarrow{OC} = p\overrightarrow{OA} + q\overrightarrow{OB}$$

and we saying that pair (p, q) is coordinates of $\overrightarrow{OC}$ in the basis $\left(\overrightarrow{OA}, \overrightarrow{OB}\right)$ and $\overrightarrow{OC}$ is linear combination of $\overrightarrow{OA}$ and $\overrightarrow{OB}$ with coefficients p and q.Also note that point C belong to the segment AB iff $\overrightarrow{OC}$ is linear combination of vectors $\overrightarrow{OA}, \overrightarrow{OB}$ with non negative coefficients p and such that $p + q = 1$. (in that case we saying that $\overrightarrow{OC}$ is convex combination of vectors $\overrightarrow{OA}, \overrightarrow{OB}$ or that segment AB is convex combination of his ends).



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Indeed let C belong to the segment AB. If $C \in \{A, B\}$ then $\overrightarrow{AC} = k\overrightarrow{AB}$,where $k \in \{0, 1\}$ .If $C \notin \{A, B\}$ then $\overrightarrow{AC}$ is collinear with $\overrightarrow{AB}$ and directed as $\overrightarrow{AB}$,that is $\overrightarrow{AC} = k\overrightarrow{AB}$ for some positive k.Hence,

$$\left\|\overrightarrow{AC}\right\| = \left\|k\overrightarrow{AB}\right\| = k\left\|\overrightarrow{AB}\right\| \iff k = \frac{\left\|\overrightarrow{AC}\right\|}{\left\|\overrightarrow{AB}\right\|} < 1.$$

Thus, if C belong to the segment AB then $\overrightarrow{AC} = k\overrightarrow{AB}$ with $k \in [0, 1]$ and since

$$\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC} = \overrightarrow{OC} - \overrightarrow{OA}, \overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = \overrightarrow{OB} - \overrightarrow{OA}$$

then

$$\overrightarrow{AC} = k\overrightarrow{AB} \iff \overrightarrow{OC} - \overrightarrow{OA} = k\left(\overrightarrow{OB} - \overrightarrow{OA}\right) \iff$$

$$\iff \overrightarrow{OC} = k\overrightarrow{OB} - k\overrightarrow{OA} + \overrightarrow{OA} \iff$$

$$\overrightarrow{OC} = (1-k)\overrightarrow{OA} + k\overrightarrow{OA} \iff \overrightarrow{OC} = p\overrightarrow{OA} + q\overrightarrow{OA},$$

where $p := 1 - k, q := k$,that is $p, q \geq 0$ and $p + q = 1$.

Opposite, let

$$\overrightarrow{OC} = p\overrightarrow{OA} + q\overrightarrow{OB},$$

where $p + q = 1$ and $p, q \geq 0$. Then, by reversing transformation above we obtain $\overrightarrow{AC} = q\overrightarrow{AB}, q \in [0, 1]$. and since

$$\overrightarrow{CB} = \overrightarrow{CA} + \overrightarrow{AB} = \overrightarrow{AB} - \overrightarrow{AC} = \overrightarrow{AB} - q\overrightarrow{AB} = (1-q)\overrightarrow{AB}$$

we obtain

$$\left\|\overrightarrow{AC}\right\| = q\left\|\overrightarrow{AB}\right\|, \left\|\overrightarrow{CB}\right\| = (1-q)\left\|\overrightarrow{AB}\right\|$$

Therefore,

$$\left\|\overrightarrow{AB}\right\| = \left\|\overrightarrow{AC}\right\| + \left\|\overrightarrow{CB}\right\| \iff C$$

belong to the segment AB.

Another variant:

Let $a := \overrightarrow{OA}, b := \overrightarrow{OB}$ and $c := \overrightarrow{OC}$. Note that $C \in AB$ iff $c - a$ is collinear to $b - a$, that is $c - a = k(b - a)$ for some real k and

$|AC| + |CB| = |AB|$, that is $\|c-a\| + \|b-c\| = \|b-a\|$ .Thus,

$$C \in AB \iff \begin{cases} c-a = k(b-a) \\ \|c-a\| + \|b-c\| = \|b-a\| \end{cases}$$

Since

$$b-c = b-a-(c-a) = b-a-k(b-a) = (1-k)(b-a)$$

then

$$\|c-a\| + \|b-c\| =$$

$$= \|b-a\| \iff \|k(b-a)\| + \|(1-k)(b-a)\| = \|b-a\| \iff$$

$$|k| \|(b-a)\| + |(1-k)| \|(b-a)\| =$$

$$= \|b-a\| \iff |k| + |(1-k)| = 1 \iff 0 \le k \le 1.$$

Hence, $C \in AB \iff c-a = k(b-a) \iff c = a(1-k) + kb$, where $k \in [0,1]$.

Barycentric coordinates.

Let A, B, C be vertices of non-degenerate triangle. Then, since $\overrightarrow{AB}$ and $\overrightarrow{AC}$ non-colinear, then for each point P on plain we have unique representation $\overrightarrow{AP} = k\overrightarrow{AB} + l\overrightarrow{AC}$,where $k, l \in \mathbb{R}$. Let O be a any point fixed on the plain. Then since

$$\overrightarrow{AP} = \overrightarrow{AO} + \overrightarrow{OP}, \overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB},\ \overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$$

we obtain

$$\overrightarrow{AO} + \overrightarrow{OP} = k\left(\overrightarrow{AO} + \overrightarrow{OB}\right) + l\left(\overrightarrow{AO} + \overrightarrow{OC}\right) \iff \overrightarrow{OP} =$$

$$= (1-k-l)\overrightarrow{OA} + k\overrightarrow{OB} + l\overrightarrow{OC}.$$

Denote $p_a := 1-k-l, p_b := k, p_c := l$, then $p_a + p_b + p_c = 1$ and

$$\overrightarrow{OP} = p_a\overrightarrow{OA} + p_b\overrightarrow{OB} + p_c\overrightarrow{OC}.$$

Suppose we have another such representation

$$\overrightarrow{OP} = q_a\overrightarrow{OA} + q_b\overrightarrow{OB} + q_c\overrightarrow{OC}$$

with $q_a + q_b + q_c = 1$, then

$$\overrightarrow{AP} = p_b\overrightarrow{AB} + p_c\overrightarrow{AC} = q_b\overrightarrow{AB} + q_c\overrightarrow{AC} \implies p_b = q_b,$$

$$p_c = q_c \implies p_a = q_a$$

Since for each point P we have unique ordered triple of real numbers (p_a, p_b, p_c) which satisfy to condition

$$p_a + p_b + p_c = 1$$

and since any such ordered triple determine some point on plain, then will call such triples barycentric coordinates of point P with respect to triangle ΔABC, because in reality barycentric coordinates independent from origin O. Indeed let O_1 another origin, then

$$\overrightarrow{O_1P} = \overrightarrow{O_1O} + \overrightarrow{OP} = (p_a + p_b + p_c)\,\overrightarrow{O_1O} + p_a\overrightarrow{OA} + p_b\overrightarrow{OB} + p_c\overrightarrow{OC} =$$

$$p_a\left(\overrightarrow{O_1O} + \overrightarrow{OA}\right) + p_b\left(\overrightarrow{O_1O} + \overrightarrow{OB}\right) + p_c\left(\overrightarrow{O_1O} + \overrightarrow{OC}\right) =$$

$$= p_a\overrightarrow{O_1A} + p_b\overrightarrow{O_1B} + \overrightarrow{O} + p_c\overrightarrow{O_1C}$$

If $p_a, p_b, p_c > 0$ then P is interior point of triangle and in that case we have clear geometric interpretation of numbers p_a, p_b, p_c.Really,since

$$\overrightarrow{OP} = p_a\overrightarrow{OA} + (p_b + p_c)\left(\frac{p_b}{p_b + p_c}\overrightarrow{OB} + \frac{p_c}{p_b + p_c}\overrightarrow{OC}\right)$$

then linear combination

$$\frac{p_b}{p_b + p_c}\overrightarrow{OB} + \frac{p_c}{p_b + p_c}\overrightarrow{OC}$$

determine some point A_1 on the segment BC,such that

$$\overrightarrow{OA_1} = \frac{p_b}{p_b + p_c}\overrightarrow{OB} + \frac{p_c}{p_b + p_c}\overrightarrow{OC}\ and \overrightarrow{OP} = p_a\overrightarrow{OA} + (p_a + p_b)\,\overrightarrow{OA_1}.$$

In particularly,

$$\overrightarrow{AP} = (p_b + p_c)\,\overrightarrow{OA_1}.$$

So, P belong to the segment $AA_1$and divide it in the ratio

$$AP \div PA_1 = (p_b + p_c) \div p_a.$$

By the same way we obtain points B_1, C_1 on CA, AB, respectively, and

$$BP \div PB_1 = (p_c + p_a) \div p_b, CP \div PC_1 = (p_a + p_b) \div p_c.$$

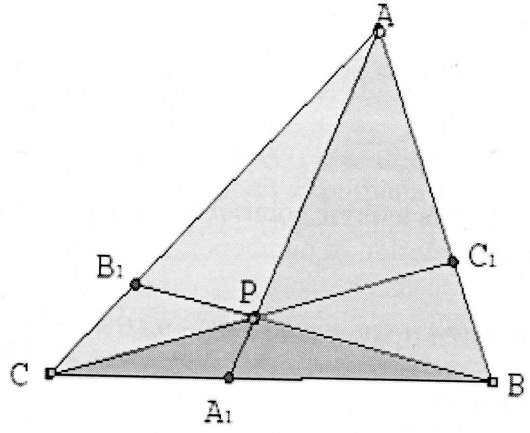
Denote

$$F_a := [PBC],\ F_b := [PCA], F_c := [PAB],\ F := [ABC]$$

then

$$p_c \div p_a = AB_1 \div CB_1 = F_c \div F_a, p_a \div p_b = BC_1 \div AC_1 = F_a \div F_b,$$

$p_b \div p_c = BC_1 \div AC_1 = F_b \div F_c$. So, $p_a \div p_b \div p_c = F_a \div F_b \div F_c$ and $p_a = \dfrac{F_a}{F}, p_b = \dfrac{F_b}{F}, p_c = \dfrac{F_c}{F}$.



Application 1. Barycentric coordinates of some triangle centres.

Problem 1. Find barycentric coordinates of the following Triangle centers:
a). Centroid G (the point of concurrency of the medians);
b). Incenter I (the point of concurrency of the interior angle bisectors);
c). Orthocenter H of an acute triangle (the point of concurrency of the altitudes);

d). Circumcenter O.

Solution.

a). Since for $P=G$ we have $F_a=F_b=F_c$ then

$$(p_a,p_b,p_c)=(1/3,1/3,1/3)$$

is barycentric coordinates of centroid G.

b). Since for $P=I$ we have

$$\frac{F_c}{F_b}=\frac{BA_1}{A_1C}=\frac{c}{b},\frac{F_a}{F_b}=\frac{BC_1}{C_1A}=\frac{a}{b}$$

then

$$F_a\div F_b\div F_c=a\div b\div c$$

and, therefore,

$$(p_a,p_b,p_c)=\frac{1}{a+b+c}(a,b,c)$$

is barycentric coordinates of incenter I.

c). For $P=H$ we have

$$BA_1=c\cos B, A_1C=b\cos C, BC_1=a\cos B, C_1A=b\cos A.$$

Hence,

$$\frac{F_c}{F_b}=\frac{BA_1}{A_1C}=\frac{c\cos B}{b\cos C}=\frac{2R\sin C\cos B}{2R\sin B\cos C}=\frac{\tan C}{\tan B},$$

$$\frac{F_a}{F_b}=\frac{BC_1}{C_1A}=\frac{a\cos B}{b\cos A}=\frac{\tan A}{\tan B}\iff F_a\div F_b\div F_c=\tan A\div\tan B\div\tan C$$

and, since

$$\frac{1}{\tan A+\tan B+\tan C}(\tan A,\tan B,\tan C)=$$

$$=\frac{1}{\tan A\tan B\tan C}(\tan A,\tan B,\tan C)=(\cot B\cot C,\cot C\cot A,\cot A\cot B),$$

then

$$(p_a,p_b,p_c)=(\cot B\cot C,\cot C\cot A,\cot A\cot B)$$

is barycentric coordinates of orthocenter H.

d). For $P = O$ since $\angle BOC = 2A$, $\angle COA = 2B, \angle AOB = 2C$ we have

$$F_a = \frac{R^2 \sin 2A}{2}, F_b = \frac{R^2 \sin 2B}{2}, F_c = \frac{R^2 \sin 2C}{2}$$

and, therefore*,

$$(p_a, p_b, p_c) = \frac{1}{\sin 2A + \sin 2B + \sin 2C}(\sin 2A, \sin 2B, \sin 2C) =$$

$$= \frac{1}{4 \sin A \sin B \sin C}(\sin 2A, \sin 2B, \sin 2C) =$$

$$\left(\frac{\cos A}{\sin B \sin C}, \frac{\cos B}{\sin C \sin A}, \frac{\cos C}{\sin A \sin B}\right)$$

is barycentric coordinates of circumcenter O.

* Note that

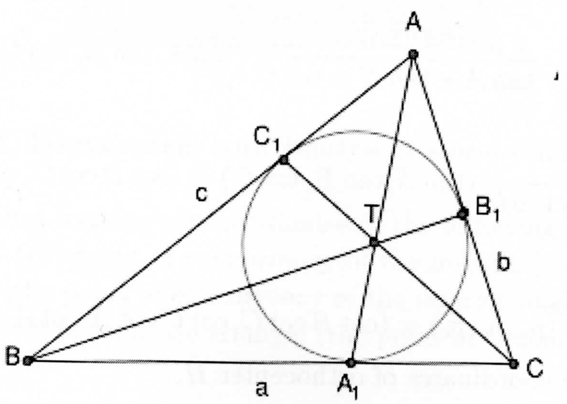
$$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$$

Problem 2.

a). Let A_1, B_1, C_1 be, respectively, points of tangency of incircle to sides BC, CA, AB of a triangle ABC. Prove that cevians AA_1, BB_1, CC_1 are intersect at one point and find barycentric coordinates of this point.

b). The same questions if A_1, B_1, C_1 be, respectively, points where excircles tangent sides BC, CA, AB.

Solution.



$AC_1 = AB_1 = s - a$, $BA_1 = BC_1 = s - b$, $CA_1 = CB_1 = s - c$,

a). Since

$$AC_1 = B_1A = s - a,\ C_1B = BA_1 = s - b,\ A_1C = CB_1 = s - c$$

then

$$\frac{BA_1}{A_1C} \cdot \frac{CB_1}{B_1A} \cdot \frac{AC_1}{C_1B} = \frac{s-b}{s-c} \cdot \frac{s-c}{s-a} \cdot \frac{s-a}{s-b} = 1$$

and, therefore, by converse of Ceva's Theorem cevians AA_1, BB_1, CC_1 are concurrent. Let T be point of intersection of these cevians. For $P = T$ we have

$$\frac{F_c}{F_b} = \frac{BA_1}{A_1C} = \frac{s-b}{s-c} = \frac{1/(s-c)}{1/(s-b)} = \frac{(s-b)(s-a)}{(s-c)(s-a)},$$

$$\frac{F_a}{F_b} = \frac{C_1B}{AC_1} = \frac{s-b}{s-a} = \frac{1/(s-a)}{1/(s-b)} = \frac{(s-b)(s-c)}{(s-c)(s-a)}.$$

Hence,

$$F_a \div F_b \div F_c = (s-b)(s-c) \div (s-c)(s-a) \div (s-a)(s-b) =$$

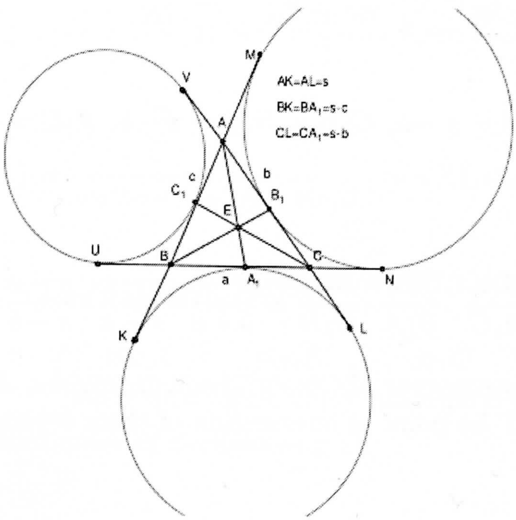
$$= \frac{1}{s-a} \div \frac{1}{s-b} \div \frac{1}{s-c}.$$

Let r_a, r_b, r_c be exradii of $\triangle ABC$. Since

$$r_a(s-a) = r_b(s-b) = r_c(s-c) = F$$

and $r_a + r_b + r_c = 4R + r$ then $F_a \div F_b \div F_c = r_a \div r_b \div r_c$ and, therefore,

$$(p_a, p_b, p_c) = \frac{1}{4R+r}(r_a, r_b, r_c)$$



b). Let $x := BA_1, y := CA_1$. Then $x + y = a$,

$$AK = AL \iff c + x = b + y$$

and, therefore,

$$2x = x + y + x - y = a + b - c \iff x = s - c,\ y = s - b$$

and $AK = AL = s$.Thus

$$BA_1 = BK = s - c, A_1C = CL = s - b.$$

Similarly, $B_1A = s - c, AC_1 = s - b$ and $BC_1 = CB_1 = s - a$. Then

$$\frac{BA_1}{A_1C} \cdot \frac{CB_1}{B_1A} \cdot \frac{AC_1}{C_1B} = \frac{s-c}{s-b} \cdot \frac{s-a}{s-c} \cdot \frac{s-b}{s-a} = 1$$

and, therefore, by converse of Ceva's Theorem cevians AA_1, BB_1, CC_1 are concurrent. Let E be point of intersection of these cevians. For $P = E$ we have

$$\frac{F_c}{F_b} = \frac{BA_1}{A_1C} = \frac{s-c}{s-b}, \frac{F_a}{F_b} = \frac{C_1B}{AC_1} = \frac{s-a}{s-b}.$$

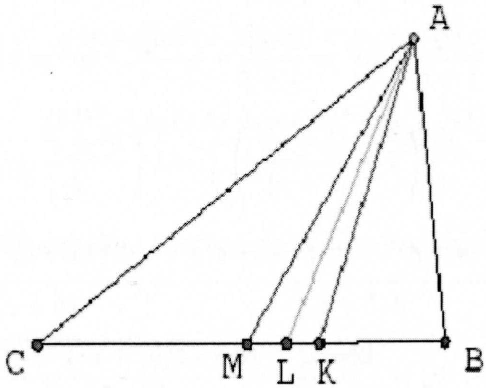
Hence,

$$F_a \div F_b \div F_c = (s-a) \div (s-b) \div (s-c)$$

and, therefore,

$$(p_a, p_b, p_c) = \frac{1}{s}(s-a, s-b, s-c).$$

Problem 3. Find barycentric coordinates of **Lemoine point** (point of intersection of symmedians).($A-$symmedian of triangle ABC is the reflection of the $A-$median in the $A-$internal angle bisector).



Pic.1

Let AM, AL, AK be respectively median, angle-bisector and symmedian of $\triangle ABC$ and let $a := BC, b := CA, c := AB, m_a := AM, w_a := AL, k_a := AK, p := ML, q := KL$. Suppose also, that $b \geq c$. Since AL v is symmedian in $\triangle ABC$ then AL is angle-bisector in triangle MAK and that imply $\dfrac{m_a}{p} = \dfrac{k_a}{q}$,i.e. there is $t > 0$ such that $k_a = tm_a$ and $q = tp$.Applying Stewart's Formula to chevian AL in triangle MAK we obtain:

$$w_a^2 = m_a^2 \cdot \frac{q}{p+q} + k_a^2 \cdot \frac{p}{p+q} - (p+q)^2 \cdot \frac{pq}{(p+q)^2} =$$

$$= m_a^2 \cdot \frac{q}{p+q} + k_a^2 \cdot \frac{p}{p+q} - pq = \frac{tm_a^2}{1+t} + \frac{k_a^2}{1+t} - tp^2,$$

because

$$\frac{p}{p+q} = \frac{1}{1+t}, \frac{q}{p+q} = \frac{t}{1+t}.$$

Since AL angle-bisector in $\triangle ABC$ then $CL = \dfrac{ab}{b+c}$ and $p = \dfrac{ab}{b+c} - \dfrac{a}{2} = \dfrac{a(b-c)}{2(b+c)}$. By substitution

$$w_a^2 = \frac{bc\left((b+c)^2 - a^2\right)}{(b+c)^2},\ m_a^2 = \frac{2\left(b^2+c^2\right) - a^2}{4}, p = \frac{a(b-c)}{2(b+c)}$$

and $k_a = tm_a$ in $w_a^2 = \dfrac{tm_a^2}{1+t} + \dfrac{k_a^2}{1+t} - tp^2$ we obtain:

$$\frac{tm_a^2}{1+t} + \frac{k_a^2}{1+t} - tp^2 = \frac{tm_a^2}{1+t} + \frac{t^2m_a^2}{1+t} - tp^2 = t\left(m_a^2 - p^2\right) =$$

$$t\left(\frac{b^2+c^2}{2} - \frac{a^2}{4}\left(1 + \frac{(b-c)^2}{(b+c)^2}\right)\right) = t\left(\frac{b^2+c^2}{2} - \frac{a^2\left(b^2+c^2\right)}{2(b+c)^2}\right) =$$

$$\frac{t\left((b+c)^2 - a^2\right)\left(b^2+c^2\right)}{2(b+c)^2} = \frac{bc\left((b+c)^2 - a^2\right)}{(b+c)^2}.$$

Hence, $t = \dfrac{2bc}{b^2+c^2},\ k_a = \dfrac{2bcm_a}{b^2+c^2} = \dfrac{bc\sqrt{2\left(b^2+c^2\right) - a^2}}{b^2+c^2}$,

$$p + q = \frac{a(b-c)}{2(b+c)}(1+t) = \frac{a(b-c)}{2(b+c)} \cdot \frac{(b+c)^2}{b^2+c^2} = \frac{a\left(b^2-c^2\right)}{2\left(b^2+c^2\right)}$$

and

$$\frac{CK}{KB} = \frac{\frac{a}{2} + p + q}{\frac{a}{2} - (p+q)} = \frac{b^2}{c^2}.$$

So, if L is Lemoin's Point (point of intersection of symmedians of $\triangle ABC$) then for barycentric coordinates (L_a, L_b, L_c) of L holds

$$L_a \div L_b \div L_c = a^2 \div b^2 \div c^2.$$

Distances Formulas.

1. Stewart's Formula for length of chevian. Let

$$\overrightarrow{OP} = p_a\overrightarrow{OA} + p_b\overrightarrow{OB}\ , p_a + p_b = 1,$$

then

$$OP^2 = \overrightarrow{OP} \cdot \overrightarrow{OP} =$$

$$\left(p_a\overrightarrow{OA} + p_b\overrightarrow{OB}\right) \cdot \left(p_a\overrightarrow{OA} + p_b\overrightarrow{OB}\right) = p_a^2OA^2 + p_b^2OB^2 + 2p_ap_b\left(\overrightarrow{OA} \cdot \overrightarrow{OB}\right) =$$

$$p_a(1-p_b)OA^2 + p_b(1-p_a)OB^2 + 2p_ap_b\left(\overrightarrow{OA}\cdot\overrightarrow{OB}\right) =$$

$$p_aOA^2 + p_bOB^2 - p_ap_bOA^2 - p_ap_bOB^2 + 2p_ap_b\left(\overrightarrow{OA}\cdot\overrightarrow{OB}\right) =$$

$$= p_aOA^2 + p_bOB^2 - p_ap_bAB^2.$$

So,

$$OP^2 = p_aOA^2 + p_bOB^2 - p_ap_bAB^2.$$

(Stewart's Formula).

2. Lagrange's Formula. Let (p_a, p_b, p_c) be baycentric coordinates of the point P, i.e

$$.p_a + p_b + p_c = 1$$

and

$$\overrightarrow{OP} = p_a\overrightarrow{OA} + p_b\overrightarrow{OB} + p_c\overrightarrow{OC},$$

then

$$OP^2 = \overrightarrow{OP}\cdot\overrightarrow{OP} = \left(p_a\overrightarrow{OA} + p_b\overrightarrow{OB} + p_c\overrightarrow{OC}\right)\cdot\overrightarrow{OP} =$$

$$p_a\overrightarrow{OA}\cdot\overrightarrow{OP} + p_b\overrightarrow{OB}\cdot\overrightarrow{OP} + p_c\overrightarrow{OC}\cdot\overrightarrow{OP} =$$

$$p_a\overrightarrow{OA}\cdot\left(\overrightarrow{OA}+\overrightarrow{AP}\right) + p_b\overrightarrow{OB}\cdot\left(\overrightarrow{OB}+\overrightarrow{BP}\right) + p_c\overrightarrow{OC}\cdot\left(\overrightarrow{OC}+\overrightarrow{CP}\right) =$$

$$\sum_{cyc}\left(p_aOA^2 + p_a\overrightarrow{OA}\cdot\overrightarrow{AP}\right) = \sum_{cyc}p_aOA^2 + \sum_{cyc}p_a\left(\overrightarrow{OP}+\overrightarrow{PA}\right)\cdot\overrightarrow{AP} =$$

$$\sum_{cyc}p_aOA^2 + \sum_{cyc}p_a\left(\overrightarrow{OP}-\overrightarrow{AP}\right)\cdot\overrightarrow{AP} = \sum_{cyc}p_a\left(OA^2 - PA^2\right) + \sum_{cyc}p_a\overrightarrow{OP}\cdot\overrightarrow{AP} =$$

$$\sum_{cyc}p_a\left(OA^2 - PA^2\right) + \overrightarrow{OP}\cdot\sum_{cyc}p_a\overrightarrow{AP} = \sum_{cyc}p_a\left(OA^2 - PA^2\right)$$

So,

$$OP^2 = \sum_{cyc}p_a\left(OA^2 - PA^2\right)$$

(Lagrange's formula).

Remark. *As a corollary from Lagrange's formula we obtain two identities which can be useful.*

Let P and be two points on plane with barycentric coordinates (p_a, p_b, p_c) and $Q\,(q_a, q_b, q_c)$, respectively. Since

$$QP^2 = \sum_{cyc} p_a \left(QA^2 - PA^2\right)$$

and $PQ^2 = \sum_{cyc} q_a \left(PA^2 - QA^2\right)$ we obtain

$$PQ^2 = \frac{1}{2}\sum_{cyc} (p_a - q_a)\left(QA^2 - PA^2\right) \text{ and } \sum_{cyc} (p_a + q_a)\left(PA^2 - QA^2\right) = 0.$$

3. Leibnitz Formula

Let A_1, B_1, C_1 be points intersection of lines PA, PB, PC with BC, CA, AB respectively. Applying Stewart Formula to $O = A_1, P$ and B, C and taking in account that

$$BA_1 \div CA_1 = p_c \div p_b$$

we obtain

$$A_1P^2 = \frac{p_b}{p_b + p_c} PB^2 + \frac{p_c}{p_b + p_c} PC^2 - \frac{p_b}{p_b + p_c} \cdot \frac{p_c}{p_b + p_c} a^2$$

and, and since

$$\overrightarrow{A_1P} = -\frac{p_a}{p_b + p_c} \overrightarrow{AP}$$

then

$$A_1P^2 = \frac{p_a^2}{(p_b + p_c)^2} AP^2.$$

Therefore,

$$\frac{p_a^2}{(p_b + p_c)^2} AP^2 = \frac{p_b}{p_b + p_c} PB^2 + \frac{p_c}{p_b + p_c} PC^2 - \frac{p_b}{p_b + p_c} \cdot \frac{p_c}{p_b + p_c} a^2 \iff$$

$$\iff p_a^2 AP^2 = p_b (p_b + p_c) PB^2 + p_c (p_b + p_c) PC^2 - p_b p_c a^2.$$

Hence,

$$\sum_{cyc} p_a^2 AP^2 = \sum_{cyc} p_b (p_b + p_c) PB^2 + \sum_{cyc} p_c (p_b + p_c) PC^2 - \sum_{cyc} p_b p_c a^2 \iff$$

$$\sum_{cyc} p_b p_c a^2 = \sum_{cyc} \left(p_b^2 + p_b p_c\right) PB^2 + \sum_{cyc} \left(p_b p_c + p_c^2\right) PC^2 - \sum_{cyc} p_a^2 AP^2 =$$

$$\sum_{cyc} p_b^2 PB^2 + \sum_{cyc} p_b p_c PB^2 + \sum_{cyc} p_b p_c PC^2 + \sum_{cyc} p_c^2 PC^2 - \sum_{cyc} p_a^2 AP^2 =$$

$$\sum_{cyc} p_b p_c PB^2 + \sum_{cyc} p_b p_c PC^2 + \sum_{cyc} p_c^2 PC^2 =$$

$$= \sum_{cyc} p_b p_c PB^2 + \sum_{cyc} p_c p_a PA^2 + \sum_{cyc} p_c^2 PC^2 =$$

$$\sum_{cyc} p_c \left(p_b PB^2 + p_a PA^2 + p_c PC^2\right) =$$

$$= \left(p_b PB^2 + p_a PA^2 + p_c PC^2\right) \sum_{cyc} p_c = \sum_{cyc} p_a PA^2$$

Thus,

$$\sum_{cyc} p_a PA^2 = \sum_{cyc} p_b p_c a^2$$

and, therefore,

$$OP^2 = \sum_{cyc} p_a \left(OA^2 - PA^2\right) \iff$$

$$OP^2 = \sum_{cyc} p_a OA^2 - \sum_{cyc} p_b p_c a^2 \text{ (\textbf{Leibnitz Formula}).}$$

Application of distance formulas.

1. Distance between circumcenter O and centroid G. Let O be circumcenter, $R-$circumradius and $P = G\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, then

$$OG^2 = \sum_{cyclic} \frac{1}{3} \cdot \left(R^2 - GA^2\right) = R^2 - \frac{1}{3} \sum_{cyclic} GA^2.$$

Since

$$GA^2 = \frac{4}{9}\left(\frac{2\left(b^2+c^2\right)-a^2}{4}\right) = \frac{2\left(b^2+c^2\right)-a^2}{9}$$

then

$$\sum_{cyclic} GA^2 = \frac{a^2+b^2+c^2}{3}$$

and

$$OG^2 = R^2 - \frac{a^2+b^2+c^2}{9}.$$

This imply

$$R^2 - \frac{a^2+b^2+c^2}{9} \geq 0 \iff a^2+b^2+c^2 \leq 9R^2.$$

2. Distance between circumcenter O and incenter I. (Euler's formula and Euler's inequality). Let O be circumcenter. Since

$$I\left(\frac{a}{a+b+c}, \frac{b}{a+b+c}, \frac{c}{a+b+c}\right),$$

then

$$(a+b+c)\,OI^2 = \sum_{cyc} a\left(OA^2 - IA^2\right) =$$

$$= \sum_{cyc} a\left(R^2 - IA^2\right) = (a+b+c)\,R^2 - \sum_{cyc} aIA^2.$$

Since

$$aIA^2 = \frac{aw_a^2\,(b+c)^2}{(a+b+c)^2} =$$

$$= \frac{abc\,(a+b+c)\,(b+c-a)\,(b+c)^2}{(a+b+c)^2\,(b+c)^2} = \frac{abc\,(b+c-a)}{a+b+c}$$

then

$$\sum_{cyclic} aIA^2 = abc$$

and

$$OI^2 = R^2 - \frac{abc}{a+b+c} = R^2 - \frac{4Rrs}{2s} = R^2 - 2Rr.$$

Hence, $OI = \sqrt{R^2 - 2Rr}$ and $R^2 - 2Rr \geq 0 \iff R \geq 2r$.

Remark. *Consider now general situation, when O be circumcenter, R−circumradius of circumcircle of △ABC and (p_a, p_b, p_c) is barycentric coordinates of some point P. Then applying general Leibnitz Formula for such origin O we obtain:*

$$OP^2 = \sum_{cyc} p_a OA^2 - \sum_{cyc} p_b p_c a^2 = \sum_{cyc} p_a R^2 - \sum_{cyc} p_b p_c a^2 = R^2 - \sum_{cyc} p_b p_c a^2.$$

Thus

$$\sum_{cyc} p_b p_c a^2 \leq R^2$$

and

$$OP = \sqrt{R^2 - \sum_{cyc} p_b p_c a^2}.$$

Using the formula obtained for the OP, we consider several more cases of calculating the distances between circumcenter O and another triangle centers..

But for beginning we will apply this formula for considered above two cases.

If $P = G\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ then

$$\sum_{cyc} p_b p_c a^2 = \frac{1}{9} \sum_{cyc} a^2$$

and, therefore,

$$OG = \sqrt{R^2 - \frac{a^2 + b^2 + c^2}{9}}$$

If $P = I\left(\frac{a}{2s}, \frac{b}{2s}, \frac{c}{2s}\right)$ then

$$\sum_{cyc} p_b p_c a^2 = \frac{1}{4s^2} \sum_{cyc} bca^2 = \frac{abc\,(a+b+c)}{4s^2} = \frac{4Rrs \cdot 2s}{4s^2} = 2Rr$$

and, therefore,

$$OI = \sqrt{R^2 - 2Rr}$$

3. Distance between incenter I and centroid G. Since

$$IA=\frac{s-a}{\cos\frac{A}{2}}$$

and

$$a^2=(b+c)^2-4bc\cos^2\frac{A}{2}\iff\cos^2\frac{A}{2}=\frac{s(s-a)}{bc}$$

then

$$IA^2=\frac{bc(s-a)}{s}.$$

By replacing O and P in **Lagrange's formula,** respectively, with I and $G\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$ and noting that

$$ab+bc+ca=s^2+4Rr+r^2, a^2+b^2+c^2=2\left(s^2-4Rr-r^2\right),$$

$abc=4Rrs$ we obtain

$$IG^2=\sum_{cyc}\frac{1}{3}\left(IA^2-GA^2\right)=\frac{1}{3}\sum_{cyc}\left(\frac{bc(s-a)}{s}-\frac{2\left(b^2+c^2\right)-a^2}{9}\right)=$$

$$=\frac{1}{3}\sum_{cyc}\frac{bc(s-a)}{s}-\frac{1}{27}\sum_{cyc}\left(2\left(b^2+c^2\right)-a^2\right)=$$

$$=\frac{s(ab+bc+ca)-3abc}{3s}-\frac{3\left(a^2+b^2+c^2\right)}{27}=$$

$$=\frac{s\left(s^2+4Rr+r^2\right)-12Rrs}{3s}-\frac{2\left(s^2-4Rr-r^2\right)}{9}=\frac{s^2-16Rr+5r^2}{9}$$

Thus,

$$s^2-16Rr+5r^2\geq 0\iff s^2\geq 16Rr-5r^2 \text{ (2-nd Gerretsen's Inequality)}$$

and

$$IG=\frac{\sqrt{s^2-16Rr+5r^2}}{3}$$

4. Distance between incenter I and orthocenter H. Since

$$HA = 2R\cos A$$

then

$$HA^2 = 4R^2\left(1 - \sin^2 A\right) = 4R^2 - a^2.$$

Also note that

$$a^3 + b^3 + c^3 = (a+b+c)^3 + 3abc - 3(a+b+c)(ab+bc+ca) =$$

$$= 8s^3 + 12Rrs - 6s\left(s^2 + 4Rr + r^2\right) = 2s\left(s^2 - 6Rr - 3r^2\right)$$

By replacing O and P in **Lagrange's formula,** respectively, with H and $I\left(\frac{a}{2s}, \frac{b}{2s}, \frac{c}{22s}\right)$ we obtain

$$HI^2 = \sum_{cyc} \frac{a}{2s}\left(HA^2 - IA^2\right) = \frac{1}{2s}\sum_{cyc}\left(a\left(4R^2 - a^2\right) - \frac{abc\,(s-a)}{s}\right) =$$

$$= \frac{1}{2s}\left(4R^2\sum_{cyc} a - \sum_{cyc} a^3 - \frac{abc}{s}\sum_{cyc}(s-a)\right) =$$

$$= \frac{1}{2s}\left(8R^2 s - 2s\left(s^2 - 6Rr - 3r^2\right) - 4Rrs\right) = 4R^2 + 4Rr + 3r^2 - s^2.$$

Thus,

$$4R^2+4Rr+3r^2-s^2 \geq 0 \iff s^2 \leq 4R^2+4Rr+3r^2 \text{ (1-st Gerretsen's Inequality)}$$

and

$$HI = \sqrt{4R^2 + 4Rr + 3r^2 - s^2}$$

5. Distance between circumcenter O and orthocenter H. Since

$$H\left(\cot B \cot C, \cot C \cot A, \cot A \cot B\right)$$

then

$$\sum_{cyc} p_b p_c a^2 = \sum_{cyc} \cot C \cot A \cdot \cot A \cot B \cdot a^2 = \cot A \cot B \cot C \sum_{cyc} a^2 \cot A.$$

Noting that

$$\sum_{cyc} \cot A \cdot a^2 =$$

$$= 4R^2 \sum_{cyc} \cot A \cdot \sin^2 A = 2R^2 \sum_{cyc} \sin 2A = 8R^2 \sin A \sin B \sin C$$

and

$$\cos A \cos B \cos C = \frac{s^2 - (2R+r)^2}{4R^2}$$

we obtain

$$\sum_{cyc} p_b p_c a^2 = \cot A \cot B \cot C \sum_{cyc} a^2 \cot A =$$

$$= \cot A \cot B \cot C \cdot 8R^2 \sin A \sin B \sin C = 8R^2 \cos A \cos B \cos C =$$

$$= 8R^2 \cdot \frac{s^2 - (2R+r)^2}{4R^2} = 2\left(s^2 - (2R+r)^2\right)$$

and, therefore,

$$OH = \sqrt{R^2 - 2\left(s^2 - (2R+r)^2\right)} = \sqrt{9R^2 + 8Rr + 2r^2 - 2s^2}.$$

And by the way we obtain inequality

$$s^2 \leq \frac{9R^2 + 8Rr + 2r^2}{2}.$$

Remark. *This inequality also immediately follows from Gerretsen's Inequality $s^2 \leq 4R^2 + 4Rr + 3r^2$ and Euler's Inequality $R \geq 2r$. Indeed,*

$$9R^2 + 8Rr + 2r^2 - 2s^2 \geq 9R^2 + 8Rr + 2r^2 - 2\left(4R^2 + 4Rr + 3r^2\right) 7 =$$

$$= (R - 2r)(R + 2r)$$

6. Distance between circumcenter O and point T.(see Problem 2a. in Application1)

Since for $P = T$ we have

$$(p_a, p_b, p_c) = \left(\frac{1}{k(s-a)}, \frac{1}{k(s-b)}, \frac{1}{k(s-c)}\right),$$

where $k = \sum_{cyc} \frac{1}{s-a} = \frac{4R+r}{sr}$ then*

$$\sum_{cyc} p_b p_c a^2 = \frac{1}{k^2} \sum_{cyc} \frac{a^2}{(s-b)(s-c)} =$$

$$= \frac{s^2 r^2}{(4R+r)^2 (s-a)(s-b)(s-c)} \sum_{cyc} a^2 (s-a) =$$

$$= \frac{s^2 r^2}{(4R+r)^2 sr^2} \sum_{cyc} a^2 (s-a) = \frac{s}{(4R+r)^2} \sum_{cyc} a^2 (s-a) = \frac{4s^2 r (R+r)}{(4R+r)^2}$$

and, therefore,

$$OT = \sqrt{R^2 - \frac{4s^2 r (R+r)}{(4R+r)^2}}.$$

And by the way we obtain inequality

$$s^2 \leq \frac{R^2 (4R+r)^2}{4r((R+r))},$$

which also can be proved using Gerretsen's Inequality $s^2 \leq 4R^2 + 4Rr + 3r^2$ and Euler's Inequality $R \geq 2r$.

* Since

$$ab + bc + ca = s^2 + 4Rr + r^2,$$

$$a^2 + b^2 + c^2 = 4s^2 - 2(ab + bc + ca) = 2\left(s^2 - 4Rr - r^2\right),$$

$$a^3 + b^3 + c^3 = 3abc + (a+b+c)^3 - 3(a+b+c)(ab+bc+ca) =$$

$$= 3 \cdot 4Rrs + 8s^3 - 6s\left(s^2 + 4Rr + r^2\right) = 2s\left(s^2 - 6Rr - 3r^2\right)$$

we obtain

$$\sum_{cyc} a^2 (s-a) = 2s\left(s^2 - 4Rr - r^2\right) - 2s\left(s^2 - 6Rr - 3r^2\right) = 4rs\,(R+r)$$

7. Distance between circumcenter O and point E (see Problem 2b. in Application1)

Since for $P = E$ we have

$$(p_a, p_b, p_c) = \frac{1}{s}(s-a, s-b, s-c)$$

then

$$\sum_{cyc} p_b p_c a^2 = \frac{1}{s^2}\sum_{cyc}(s-b)(s-c)\,a^2 =$$

$$= \frac{1}{s^2}\sum_{cyc}\left(a^2 s^2 - a^2 s\,(b+c) + a^2 bc\right) =$$

$$= a^2 + b^2 + c^2 + \frac{abc\,(a+b+c)}{s^2} - \frac{(a+b+c)(ab+bc+ca)}{s} + \frac{3abc}{s} =$$

$$= 2\left(s^2 - 4Rr - r^2\right) + 8Rr - 2\left(s^2 + 4Rr + r^2\right) + 12Rr = 4r\,(R-r)$$

and,therefore,

$$OE = \sqrt{R^2 - 4r\,(R-r)} = R - 2r$$

and, by the way, our calculation of QE give us one more proof of Euler's Inequality.

8. Distance between circumcenter O and point L (Lemioin's point).

Since for $P = L$ we have

$$(p_a, p_b, p_c) = \frac{1}{a^2+b^2+c^2}\left(a^2, b^2, c^2\right)$$

then

$$\sum_{cyc} p_b p_c a^2 = \frac{1}{(a^2+b^2+c^2)^2}\sum_{cyc}$$

$$b^2c^2 \cdot a^2 = \frac{3a^2b^2c^2}{\left(a^2+b^2+c^2\right)^2}$$

and, therefore,

$$OL = \sqrt{R^2 - \frac{3a^2b^2c^2}{\left(a^2+b^2+c^2\right)^2}} =$$

$$= \sqrt{R^2 - \frac{48R^2r^2s^2}{\left(a^2+b^2+c^2\right)^2}} = R\sqrt{1 - \frac{48F^2}{\left(a^2+b^2+c^2\right)^2}}$$

and, by the way, our calculation of QL give us one more proof of Weitzenböck's inequality

$$a^2+b^2+c^2 \geq 4\sqrt{3}F.$$

Remark. Since

$$\left(a^2+b^2+c^2\right)^2 - 48F^2 =$$

$$= \left(a^2+b^2+c^2\right)^2 - 3\left(2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4\right) =$$

$$= 4\left(a^4+b^4+c^4-a^2b^2-a^2c^2-b^2c^2\right)$$

then

$$OL = 2R\sqrt{\frac{a^4+b^4+c^4-a^2b^2-a^2c^2-b^2c^2}{\left(a^2+b^2+c^2\right)^2}}$$

Problem 4.

Let ABC be a triangle with sidelengths a, b, c and let M be any point lying on circumcircle of $\triangle ABC$.v Find the maximum and minimum of the the following expression:

a). $a \cdot MA^2 + b \cdot MB^2 + c \cdot MC^2$(All Israel Math Olympiad);.
b). $\tan A \cdot MA^2 + \tan B \cdot MB^2 + \tan C \cdot MC^2$ if $\triangle ABC$ is acute angled triangle;
c). $\sin 2A \cdot MA^2 + \sin 2B \cdot MB^2 + \sin 2C \cdot MC^2$;
d). $a^2 \cdot MA^2 + b^2 \cdot MB^2 + c^2 \cdot MC^2$;
e). $\dfrac{MA^2}{s-a} + \dfrac{MB^2}{s-b} + \dfrac{MC^2}{s-c}$.

f). $(s-a)\,MA^2+(s-b)\,MB^2+\ (s-c)\,MC^2$

Solution. First we consider a common approach to the all these problems represented in the following general formulation:

Let α,β,γ be real numbers such that $\alpha+\beta+\gamma\neq 0$ and let M be any point lying on circumcircle of a triangle ABC with sidelengths a,b,c and circumradius R
Find the maximal and the minimal values of the expression:

$$D(M):=\ \alpha\cdot MA^2+\beta\cdot MB^2+\gamma\cdot MC^2.$$

Let P be a point on the plane with barycentric coordinates

$$(p_a,p_b,p_c)=\frac{1}{\alpha+\beta+\gamma}(\alpha,\beta,\gamma\ ).$$

Then, by replacing origin O in the Leibnitz Formula with $M,$ we obtain

$$MP^2=\sum_{cyc}p_aMA^2-\sum_{cyc}p_bp_ca^2=$$

$$=\frac{1}{\alpha+\beta+\gamma}\sum_{cyc}\alpha\cdot MA^2-\frac{1}{(\alpha+\beta+\gamma)^2}\sum_{cyc}\beta\gamma a^2\iff$$

$$D(M)=(\alpha+\beta+\gamma)\,MP^2+\frac{1}{\alpha+\beta+\gamma}\sum_{cyc}\beta\gamma a^2=$$

$$=(\alpha+\beta+\gamma)\left(MP^2+\sum_{cyc}p_bp_ca^2\right).$$

Since $\sum\limits_{cyc}p_bp_ca^2$ isn't depend from M then the problem reduces to finding the largest and smallest value of $(\alpha+\beta+\gamma)\,MP^2.$ Wherein if $\alpha+\beta+\gamma<0$ then

$$\max\left((\alpha+\beta+\gamma)\,MP^2\right)=(\alpha+\beta+\gamma)\min MP^2$$

and

$$\min\left((\alpha+\beta+\gamma)\,MP^2\right)=(\alpha+\beta+\gamma)\max MP^2.$$

Bearing in mind the application of the general case to the problems listed above, and also not to overload the text, we assume further that

$\alpha+\beta+\gamma>0$ and that point P is interior with respect to circumcircle. Then if d is the distant between point P and circumcenter O then $\max MP = R+d$ and $\min MP = R-d$.

$$\max D(M) = (\alpha+\beta+\gamma)\left((R+d)^2+\sum_{cyc} p_b p_c a^2\right)$$

and

$$\min D(M) = (\alpha+\beta+\gamma)\left((R-d)^2+\sum_{cyc} p_b p_c a^2\right).$$

Coming back to the listed above subproblems we obtain:

a). Since

$$(\alpha,\beta,\gamma) = (a,b,c), P = I, (p_a,p_b,p_c) = \left(\frac{a}{2s},\frac{b}{2s},\frac{c}{2c}\right),$$

$$d = OI = \sqrt{R^2-2Rr}$$

and

$$\sum_{cyc} p_b p_c a^2 = 2Rr$$

(see **Distance between circumcenter O and incenter I**) then for

$$D(M) = a\cdot MA^2 + b\cdot MB^2 + c\cdot MC^2$$

we obtain

$$\max D(M) = (a+b+c)\left(\left(R+\sqrt{R^2-2Rr}\right)^2+2Rr\right) =$$

$$= 4Rs\left(R+\sqrt{R^2-2Rr}\right)$$

and

$$\min D(M) = (a+b+c)\left(\left(R-\sqrt{R^2-2Rr}\right)^2+2Rr\right) =$$

$$= 4Rs\left(R-\sqrt{R^2-2Rr}\right).$$

b). Since

$$(\alpha,\beta,\gamma) = (\tan A,\tan B,\tan C),\ (p_a,p_b,p_c) =$$

$$= (\cot B \cot C, \cot C \cot A, \cot A \cot B),$$

$$d = OH = \sqrt{9R^2 + 8Rr + 2r^2 - 2s^2},\ \tan A + \tan B + \tan C = \frac{2sr}{s^2 - (2R+r)^2}$$

and

$$\sum_{cyc} p_b p_c a^2 = 2\left(s^2 - (2R+r)^2\right)$$

(see **Distance between circumcenter** O **and orthocenter** H) then for

$$D(M) = \tan A \cdot MA^2 + \tan B \cdot MB^2 + \tan C \cdot MC^2$$

we obtain

$$\max D(M) = (\tan A + \tan B + \tan C) \cdot$$

$$\cdot \left(\left(R + \sqrt{9R^2 + 8Rr + 2r^2 - 2s^2}\right)^2 + 2\left(s^2 - (2R+r)^2\right)\right) =$$

$$= \frac{2sr}{s^2 - (2R+r)^2} \cdot 2R\left(R + \sqrt{9R^2 + 8Rr + 2r^2 - 2s^2}\right) =$$

$$= \frac{4Rrs\left(R + \sqrt{9R^2 + 8Rr + 2r^2 - 2s^2}\right)}{s^2 - (2R+r)^2}$$

and

$$\min D(M) = \frac{4Rrs\left(R - \sqrt{9R^2 + 8Rr + 2r^2 - 2s^2}\right)}{s^2 - (2R+r)^2}$$

c). Since

$$(\alpha, \beta, \gamma) = (\sin 2A, \sin 2B, \sin 2C),\ P = O,$$

$$(p_a, p_b, p_c) = \left(\frac{\cos A}{\sin B \sin C}, \frac{\cos B}{\sin C \sin A}, \frac{\cos C}{\sin A \sin B}\right)$$

and $d = OO = 0$ then

$$D(M) = \sin 2A \cdot MA^2 + \sin 2B \cdot MB^2 + \sin 2C \cdot MC^2 =$$

$$= (\sin 2A + \sin 2B + \sin 2C) \sum_{cyc} \frac{\cos B}{\sin C \sin A} \cdot \frac{\cos C}{\sin A \sin B} a^2 =$$

$$= 4 \sin A \sin B \sin C \sum_{cyc} \frac{a^2 \cos B \cos C}{\sin^2 A \sin C \sin B} =$$

$$= 4 \sum_{cyc} \frac{a^2 \cos B \cos C}{\sin A} = 8R^2 \sum_{cyc} \sin A \cos B \cos C.$$

That is for any point M that lies on circumcircle $D(M)$ is the constant, namely

$$\sum_{cyc} \sin 2A \cdot MA^2 = 8R^2 \sum_{cyc} \sin A \cos B \cos C.$$

d). Since

$$(\alpha, \beta, \gamma) = \left(a^2, b^2, c^2\right), \; P = L, \; (p_a, p_b, p_c) = \frac{1}{a^2 + b^2 + c^2} \left(a^2, b^2, c^2\right),$$

$$d = OL = R\sqrt{1 - \frac{48F^2}{(a^2 + b^2 + c^2)^2}}, \; \sum_{cyc} p_b p_c a^2 =$$

$$= \frac{3a^2b^2c^2}{(a^2 + b^2 + c^2)^2} = \frac{48R^2F^2}{(a^2 + b^2 + c^2)^2}$$

(see **Distance between circumcenter O and Lemoin point L**) then for

$$D(M) = a^2 \cdot MA^2 + b^2 \cdot MB^2 + c^2 \cdot MC^2$$

we obtain

$$\max D(M) =$$

$$= (a^2 + b^2 + c^2) \left(R^2 \left(1 + \sqrt{1 - \frac{48F^2}{(a^2 + b^2 + c^2)^2}} \right)^2 + \frac{48R^2F^2}{(a^2 + b^2 + c^2)^2} \right) =$$

$$= \frac{R^2}{a^2 + b^2 + c^2} \left(\left(a^2 + b^2 + c^2 + \sqrt{(a^2 + b^2 + c^2)^2 - 48F^2} \right)^2 + 48F^2 \right) =$$

$$= 2R^2\left(2\sqrt{a^4+b^4+c^4-a^2b^2-a^2c^2-b^2c^2}+a^2+b^2+c^2\right)$$

because

$$\left(a^2+b^2+c^2\right)^2-48F^2=4\left(a^4+b^4+c^4-a^2b^2-a^2c^2-b^2c^2\right)$$

and

$$\left(t+\sqrt{t^2-48F^2}\right)^2+48F^2=2t\left(\sqrt{t^2-48F^2}+t\right),$$

where $t=a^2+b^2+c^2$. Also,

$$\min D(M)=$$

$$=\left(a^2+b^2+c^2\right)\left(R^2\left(1-\sqrt{1-\frac{48F^2}{\left(a^2+b^2+c^2\right)^2}}\right)^2+\frac{48R^2F^2}{\left(a^2+b^2+c^2\right)^2}\right)=$$

$$=\frac{R^2}{a^2+b^2+c^2}\left(\left(a^2+b^2+c^2-\sqrt{\left(a^2+b^2+c^2\right)^2-48F^2}\right)^2+48F^2\right)=$$

$$=2R^2\left(a^2+b^2+c^2-2\sqrt{a^4+b^4+c^4-a^2b^2-a^2c^2-b^2c^2}\right)$$

e). Since

$$(\alpha,\beta,\gamma)=\left(\frac{1}{s-a},\frac{1}{s-b},\frac{1}{s-c}\right),\quad P=T,$$

$$(p_a,p_b,p_c)=\left(\frac{1}{k(s-a)},\frac{1}{k(s-b)},\frac{1}{k(s-c)}\right),$$

where

$$k=\sum_{cyc}\frac{1}{s-a}=\frac{4R+r}{sr},\quad d=OT=\sqrt{R^2-\frac{4s^2r(R+r)}{(4R+r)^2}}$$

and

$$\sum_{cyc}p_bp_ca^2=\frac{4s^2r(R+r)}{(4R+r)^2}$$

(see **Distance between circumcenter O and T**) then for

$$D(M)=\frac{MA^2}{s-a}+\frac{MB^2}{s-b}+\frac{MC^2}{s-c}$$

we obtain

$$\max D(M)=\left(\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}\right)\cdot$$

$$\cdot\left(\left(R+\sqrt{R^2-\frac{4s^2r(R+r)}{(4R+r)^2}}\right)^2+\frac{4s^2r(R+r)}{(4R+r)^2}\right)=$$

$$=\frac{4R+r}{sr}\cdot 2R\left(R+\frac{\sqrt{R^2(4R+r)^2-4rs^2(R+r)}}{4R+r}\right)=$$

$$=\frac{2R\left(R(4R+r)+\sqrt{R^2(4R+r)^2-4rs^2(R+r)}\right)}{sr}$$

and

$$\min D(M)=\frac{2R\left(R(4R+r)-\sqrt{R^2(4R+r)^2-4rs^2(R+r)}\right)}{sr}$$

f). Since

$$(\alpha,\beta,\gamma)=(s-a,s-b,s-c),$$

$$P=E,(p_a,p_b,p_c)=\frac{1}{s}(s-a,s-b,s-c),$$

$$\sum_{cyc}p_bp_ca^2=4r(R-r),d=OE=R-2r$$

(see **Distance between circumcenter O and E**) then for

$$D(M)=(s-a)MA^2+(s-b)MB^2+(s-c)MC^2$$

we obtain

$$\max D(M)=s\left((R+R-2r)^2+4r(R-r)\right)=4sR(R-r)$$

and

$$\min D(M) = s\left((R-(R-2r))^2 + 4r(R-r)\right) = 4Rsr = abc.$$

Problem 5. Let a, b, c be sidelengths of a triangle ABC. Find point O in the plane such that the sum

$$\frac{OA^2}{b^2} + \frac{OB^2}{c^2} + \frac{OC^2}{a^2}$$

is minimal.

Solution. Let P be point on the plane with barycentric coordinates

$$(p_a, p_b, p_c) = \left(\frac{1}{kb^2}, \frac{1}{kc^2}, \frac{1}{ka^2}\right),$$

where $k = \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{a^2}$.

Then by Leibnitz Formula

$$OP^2 = \sum_{cyc} p_a OA^2 - \sum_{cyc} p_b p_c a^2 = \frac{1}{k}\sum_{cyc}\frac{OA^2}{b^2} - \frac{1}{k^2}\sum_{cyc}\frac{1}{c^2a^2}\cdot a^2 =$$

$$\frac{1}{k}\sum_{cyc}\frac{OA^2}{b^2} - \frac{1}{k^2}\sum_{cyc}\frac{1}{c^2} = \frac{1}{k}\left(\sum_{cyc}\frac{OA^2}{b^2} - 1\right).$$

Hence,

$$\sum_{cyc}\frac{OA^2}{b^2} = k\cdot OP^2 + 1$$

and, therefore,

$$\min\sum_{cyc}\frac{OA^2}{b^2} = 1 = \sum_{cyc}\frac{PA^2}{b^2}.$$

That is $\sum_{cyc}\frac{OA^2}{b^2}$ is minimal iff $O = P$, where P is intersect point of cevians AA_1, BB_1, CC_1 such that

$$\frac{BA_1}{A_1C} = \frac{F_c}{F_b} = \frac{p_c}{p_b} = \frac{c^2}{a^2}, \frac{CB_1}{B_1A} = \frac{p_a}{p_c} = \frac{a^2}{b^2}, \frac{AC_1}{C_1B} = \frac{p_b}{p_a} = \frac{b^2}{c^2}.$$

Problem 6. Let ABC be a triangle with sidelengths $a = BC, b = CA, c = AB$ and let s, R and r be semiperimeter, circumradius and inradius of $\triangle ABC$, respectively. For any point P lying on incircle of $\triangle ABC$ let

$$D(P) := aPA^2 + bPB^2 + cPC^2.$$

Prove that $D(P)$ is a constant and find its value in terms of s, R and r.

Solution. Let I be incener of $\triangle ABC$ and let (i_a, i_b, i_c) be baricentric coordinates of I. Since

$$(i_a, i_b, i_c) = \frac{1}{2s}(a, b, c)$$

and $PI = r$ then applying Leibnitz Formula for distance between points I and P we obtain

$$r^2 = PI^2 = \sum_{cyc} i_a \cdot PA^2 - \sum_{cyc} i_b i_c a^2 = \frac{1}{2s}\sum aPA^2 - \frac{1}{4s^2}\sum_{cyc} bca^2 =$$

$$= \frac{1}{2s}\sum_{cyc} aPA^2 - \frac{abc \cdot 2s}{4s^2} = \frac{1}{2s}\sum_{cyc} aPA^2 - \frac{4Rrs}{2s} = \frac{1}{2s}\sum aPA^2 - 2Rr.$$

Hence,

$$\sum_{cyc} aPA^2 = 2s\left(r^2 + 2Rr\right).$$

Area of a triangle, equation of a line and equation of a circle in barycentric coordinates.

1. **Area of a triangle.** First we recall that for any two vectors a, b on the plane is defined skew product

$$a \wedge b := \|a\| \, \|b\| \sin\left(\widehat{a, b}\right)$$

and if (a_1, a_2) , (b_1, b_2) are Cartesian coordinates of a, b,respectively, then

$$a \wedge b = \det\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} = a_1 b_2 - a_2 b_1.$$

Geometrically $a \wedge b$ is oriented (because $a \wedge b = -b \wedge a$) area of parallelogram defined by vectors a, b.Obvious that $a \wedge b = 0$ iff a, b are collinear (in particular $a \wedge a = 0$ for any a).
Using coordinate definition of skew product easy to prove that it is bilinear, that is

$$(a + b) \wedge c = a \wedge c + b \wedge c$$

then also

$$a\wedge(c + b) = -(c + b)\wedge a = -(c \wedge a + b \wedge a) = (-c \wedge a) + (-b \wedge a) = a\wedge c + a\wedge b$$

and

$$(pa) \wedge b = a \wedge (pb) = p(a \wedge b)$$

for any real p.
For any three point K, L, M on the plane which are not collinear we will use common notation $[K, L, M]$ for oriented area of $\triangle KLM$
which equal to $\frac{1}{2}\overrightarrow{KL} \wedge \overrightarrow{KM}$ (in the case if K, L, M are collinear we obtain $[K, L, M] = 0$). Regular area of $\triangle KLM$ is $\frac{1}{2}\left|\overrightarrow{KL} \wedge \overrightarrow{KM}\right|$.
Let P, Q, R be three point on the plane and $(p_a, p_b, p_c), (q_a, q_b, q_c), (r_a, r_b, r_c)$ be, respectively their barycentric coordinates with respect to triangle ABC.Then

$$\overrightarrow{AP} = p_a\overrightarrow{AA} + p_b\overrightarrow{AB} + p_c$$

and, similarly,

$$\overrightarrow{AQ} = q_b\overrightarrow{AB} + q_c\overrightarrow{AC}, \ \overrightarrow{AR} = r_b\overrightarrow{AB} + r_c\overrightarrow{AC}\ .$$

Hence,

$$\overrightarrow{PQ} = (q_b - p_b)\overrightarrow{AB} + (q_c - p_c)\overrightarrow{AC}, \ \overrightarrow{PR} = (r_b - p_b)\overrightarrow{AB} + (r_c - p_c)\overrightarrow{AC}$$

and, therefore,

$$2[P, Q, R] = \overrightarrow{PQ} \wedge \overrightarrow{PR} =$$

$$= \left((q_b - p_b)\overrightarrow{AB} + (q_c - p_c)\overrightarrow{AC}\right) \wedge \left((r_b - p_b)\overrightarrow{AB} + (r_c - p_c)\overrightarrow{AC}\right) =$$

$$= (q_b - p_b)(r_c - p_c)\,\overrightarrow{AB} \wedge \overrightarrow{AC} + (q_c - p_c)(r_b - p_b)\,\overrightarrow{AC} \wedge \overrightarrow{AB} =$$

$$((q_b - p_b)(r_c - p_c) - (r_b - p_b)(q_c - p_c))\,\overrightarrow{AB} \wedge \overrightarrow{AC} =$$

$$= 2\,[A, B, C] \cdot \det \begin{pmatrix} q_b - p_b & r_b - p_b \\ q_c - p_c & r_c - p_c \end{pmatrix}.$$

Thus,

$$[P, Q, R] = \det \begin{pmatrix} q_b - p_b & r_b - p_b \\ q_c - p_c & r_c - p_c \end{pmatrix} \cdot [A, B, C].$$

Or, since

$$\det \begin{pmatrix} q_b - p_b & r_b - p_b \\ q_c - p_c & r_c - p_c \end{pmatrix} =$$

$$= (q_b - p_b)(r_c - p_c) - (r_b - p_b)(q_c - p_c) =$$

$$= p_b q_c + p_c r_b + q_b r_c - p_c q_b - p_b r_c - q_c r_b = \det \begin{pmatrix} 1 & p_b & p_c \\ 1 & q_b & q_c \\ 1 & r_b & r_c \end{pmatrix} =$$

$$= \det \begin{pmatrix} p_a & p_b & p_c \\ q_a & q_b & q_c \\ r_a & r_b & r_c \end{pmatrix}$$

(because $1 - p_b - p_c = p_a, 1 - q_b - q_c = q_a, 1 - r_b - r_c = r_a$) and, therefore, we obtain more representative form of obtained correlation (Areas Formula)

$$\textbf{(AF)} \qquad [P, Q, R] = \det \begin{pmatrix} p_a & p_b & p_c \\ q_a & q_b & q_c \\ r_a & r_b & r_c \end{pmatrix} [A, B, C].$$

Using this formula we can to do important conclusion, namely:

Points P, Q, R are collinear iff $\det \begin{pmatrix} p_a & p_b & p_c \\ q_a & q_b & q_c \\ r_a & r_b & r_c \end{pmatrix} = 0.$

From that immediately follows that set of points on the plane with barycentric coordinates (x, y, z) such that $\det \begin{pmatrix} x & y & z \\ q_a & q_b & q_c \\ r_a & r_b & r_c \end{pmatrix} = 0$ is line which passed through points $Q(q_a, q_b, q_c)$ and $R(r_a, r_b, r_c)$,that is

$$\det\begin{pmatrix} x & y & z \\ q_a & q_b & q_c \\ r_a & r_b & r_c \end{pmatrix} = 0$$ is equation of line in baycentric coordinates.

As another application of formula (AF) we will solve the following

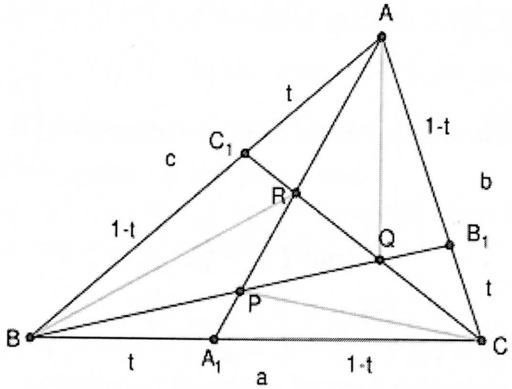
Problem 7. Let AA_1, BB_1, CC_1 be cevians of a triangle ABC such that

$$\frac{AB_1}{B_1C} = \frac{CA_1}{A_1B} = \frac{BC_1}{C_1A} = \frac{1-t}{t}.$$

Find the ratio

$$\frac{[P,Q,R]}{[A,B,C]}.$$

Solution.



Let $(p_a, p_b, p_c), (q_a, q_b, q_c), (r_a, r_b, r_c)$ be, respectively, barycentric coordinates of points P, Q, R . Then

$$\frac{A_1B}{A_1C} = \frac{t}{1-t} = \frac{p_c}{p_b}, \frac{B_1C}{B_1A} = \frac{t}{1-t} = \frac{p_a}{p_c}.$$

Noting that

$$\frac{p_a}{p_c} = \frac{t}{1-t} = \frac{t^2}{t(1-t)}, \frac{p_b}{p_c} = \frac{1-t}{t} = \frac{(1-t)^2}{t(1-t)}$$

we can conclude that

$$p_a = kt^2, p_b = k(1-t)^2,\ p_c = kt(1-t),$$

for some k and since $p_a + p_b + p_c = 1$ we obtain

$$k\left(t^2 + (1-t)^2 + t(1-t)\right) = 1 \iff k\left(t^2 - t + 1\right) =$$

$$= 1 \iff k = \frac{1}{t^2 - t + 1}.$$

Hence,

$$p_a = \frac{t^2}{t^2 - t + 1}, p_b = \frac{(1-t)^2}{t^2 - t + 1}, p_c = \frac{t(1-t)}{t^2 - t + 1}.$$

Since $\dfrac{q_c}{q_a} = \dfrac{1-t}{t}$ and $\dfrac{q_b}{q_a} = \dfrac{t}{1-t}$ we, as above, obtain

$$q_a = \frac{t(1-t)}{t^2 - t + 1} = p_c, q_b = \frac{t^2}{t^2 - t + 1} = p_a, q_c = \frac{(1-t)^2}{t^2 - t + 1} = p_b,$$

that is

$$(q_a, q_b, q_c) = (p_c, p_a, p_b)$$

and, similarly,

$$(r_a, r_b, r_c) = (p_b, p_c, p_a).$$

Hence,

$$\frac{[P,Q,R]}{[A,B,C]} = \begin{pmatrix} p_a & p_b & p_c \\ p_c & p_a & p_b \\ p_b & p_c & p_a \end{pmatrix} =$$

$$= p_a^3 + p_b^3 + p_c^3 - 3p_a p_b p_c = (p_a + p_b + p_c)^3 -$$

$$-3(p_a + p_b + p_c)(p_a p_b + p_b p_c + p_c p_a) =$$

$$= 1 - 3(p_a p_b + p_b p_c + p_c p_a) =$$

$$= \frac{1}{(t^2 - t + 1)^2}\left(t^2(1-t)^2 + (1-t)^3 t + t^3(1-t)\right) =$$

$$= \frac{t(1-t)\left(t(1-t) + (1-t)^2 + t^2\right)}{(t^2 - t + 1)^2} = \frac{t(1-t)}{t^2 - t + 1}.$$

Equation of a circle in barycentric coordinates. Let O be center of a circle with radius R. And let P be any point on lying on this circle. If (o_a, o_b, o_c) and

$$(p_a, p_b, p_c) = (x, y, z)$$

be, respectively, barycentric coordinates of O and P then by Leybnitz Formula

$$OP^2 = \sum_{cyc} p_a OA^2 - \sum_{cyc} p_b p_c a^2 \iff$$

(EC) $$R^2 = xOA^2 + yOB^2 + zOC^2 - yza^2 - zxb^2 - xyc^2.$$

In particular, if O and R be circumcenter and circumradius of $\triangle ABC$ then

$$xOA^2 + yOB^2 + zOC^2 = R^2 (x + y + z) = R^2$$

and, therefore,

(ECc) $$yza^2 + zxb^2 + xyc^2 = 0$$

is equation of circumcircle of $\triangle ABC$.

By replacing O and R in **(EC)** with I (incenter) and r (inradius) we obtain

$$r^2 = xIA^2 + yIB^2 + zIC^2 - yza^2 - zxb^2 - xyc^2.$$

Since $IA = \dfrac{b+c}{a+b+c} \cdot l_a$, where l_a is length of angle bisector from A and $l_a = \dfrac{2\sqrt{bcs(s-a)}}{b+c}$ then

$$IA^2 = \frac{(b+c)^2}{4s^2} \cdot \frac{4bcs(s-a)}{(b+c)^2} = \frac{bc(s-a)}{s}$$

and, cyclic,

$$IB^2 = \frac{ca(s-b)}{s}, IC^2 = \frac{ab(s-c)}{s}$$

Hence,

(EIc) $$r^2 s = xbc(s-a) + yca(s-b) + zab(s-c) - yza^2 -$$

$$-zxb^2 - xyc^2 \iff$$

$$\iff xbc(s-a) + yca(s-b) + zab(s-c) - yza^2 - zxb^2 - xyc^2 =$$

$$= (s-a)(s-b)(s-c)$$

is equation of incircle.

More applications to inequalities. For further we will use compact notations for R_a, R_b, R_c for AP, BP, CP respectively.

Application 1. For triangle $\triangle ABC$ with sides a, b, c and arbitrary interior point P holds inequalities:

$$\frac{a^2+b^2+c^2}{3} \leq R_a^2 + R_b^2 + R_c^2$$

Proof. Applying Lagrange's formula to the point $G\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ (medians intersection point) and point P, we obtain

$$PG^2 = \frac{1}{3}\left(PA^2 - GA^2\right) + \frac{1}{3}\left(PB^2 - GB^2\right) + \frac{1}{3}\left(PC^2 - GC^2\right) =$$

$$\frac{1}{3}\left(R_a^2 + R_b^2 + R_c^2\right) - \frac{1}{3} \cdot \frac{4}{9}\left(m_a^2 + m_b^2 + m_c^2\right) =$$

$$= \frac{1}{3}\left(R_a^2 + R_b^2 + R_c^2\right) - \frac{4}{27} \cdot \frac{3}{4}\left(a^2+b^2+c^2\right).$$

Hence,

$$PG^2 = \frac{1}{3}\left(R_a^2 + R_b^2 + R_c^2\right) - \frac{a^2+b^2+c^2}{9}$$

and that implies inequality

$$\boxed{R_a^2 + R_b^2 + R_c^2 \geq \frac{a^2+b^2+c^2}{3}}$$

with equality condition $P = G$ (centroid-medians intersection point).

Application 2. Let x, y, z be any real numbers such that $x + y + z = 1$ and, which can be taken as barycentric coordinates of some point P on plane, that is

$$(p_a, p_b, p_c) = (x, y, z).$$

Then

$$\sum_{cyc} xOA^2 - \sum_{cyc} yza^2 = OP^2 \geq 0$$

yields inequality

$$\textbf{(R)} \qquad \sum_{cyc} xR_a^2 \geq \sum_{cyc} yza^2,$$

where $R_a := OA, R_b := OB, R_c := OC$ and O is any point in the triangle $T(a,b,c)$.
In homogeneous form this inequality becomes

$$\textbf{(Rh)} \qquad \sum_{cyc} x \cdot \sum_{cyc} xR_a^2 \geq \sum_{cyc} yza^2$$

which holds for any real x, y, z.
If $x := w - v, y := u - w, z := v - u$ then $\sum_{cycc} x = 0$ and we obtain

$$0 \geq \sum_{cyc} (u-w)(v-u)a^2 \iff$$

$$\iff \sum_{cyc} a^2 (u-w)(u-v) \geq 0 \text{ (Schure kind Inequality).}$$

By replacing (x,y,z) in **(R)** with $\left(\frac{x}{R_a^2}, \frac{y}{R_b^2}, \frac{z}{R_c^2}\right)$ we obtain

$$\sum_{cyc} \frac{x}{R_a^2} \cdot \sum_{cyclic} \frac{x}{R_a^2} \cdot R_a^2 \geq \sum_{cyc} \frac{y}{R_b^2} \cdot \frac{z}{R_c^2} a^2 \iff$$

$$\textbf{(RR)} \sum_{cyc} xR_b^2R_c^2 \cdot \sum_{cyclic} x \geq \sum_{cyc} yza^2R_a^2.$$

By substitution $x = aR_a, \ y = bR_b, \ z = cR_c$ in (*) we obtain

$$\sum_{cycl} aR_aR_b^2R_c^2 \cdot \sum_{cyc} aR_a \geq$$

$$\geq \sum_{cyc} bR_bcR_ca^2R_a^2 \iff \sum_{cyc} aR_bR_c \cdot \sum_{cyc} aR_a \geq abc \cdot aR_a \iff$$

$$\textbf{(H)} \sum_{cyc} aR_bR_c \geq abc \ (\textbf{T.Hayashi inequality}).$$

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